Math 336 Sample Problems

One notebook sized page of notes (both sides) will be allowed on the test. The test will be comprehensive. The final exam is at 8:30 am, Monday, June 10, in the regular classroom.

1. Prove that
$$e^{\frac{1}{2}(z-1/z)} = \sum_{-\infty}^{+\infty} a_n z^n, z \neq 0,$$
and that
$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - \sin\theta)d\theta.$$

2. Prove that the function $u(z) = Re(\frac{i+z}{i-z})$ and $u(i) = 0$ is harmonic in the unit disk and 0 on the boundary. What is the $\lim_{z\to i} u(z)$?

3. If f is analytic and nonconstant on $\mathbb{C} \setminus \{z_1, z_2, \ldots, z_n\}$ then the image of f is dense in $\mathbb{C}$.

4. Let $p(z)$ be a polynomial of degree n. Let $M(r) = \max\{|p(z)| : |z| = r\}$. Let $r > s > 0$. Prove that
$$\frac{M(s)}{s^n} \geq \frac{M(r)}{r^n}.$$

5. Is there an analytic function f that maps $|z| < 1$ into $|z| < 1$ such that $f(\frac{1}{2}) = \frac{2}{3}, f(\frac{1}{4}) = \frac{1}{3}$?

6. Suppose u_n is a sequence of harmonic functions on a domain W and suppose the sequence converges uniformly on compact sets to a function u. Prove that u is harmonic.

7. Let $f(z) = \dfrac{z-a}{1-\bar{a}z}$, where $|a| < 1$. Let $D = \{z : |z| < 1\}$. Prove that

   (a)
$$\frac{1}{\pi}\int_D |f'(z)|^2 dxdy = 1.$$

   (b)
$$\frac{1}{\pi}\int_D |f'(z)|dxdy = \frac{1-|a|^2}{|a|^2}\log\left(\frac{1}{1-|a|^2}\right).$$

   Hint: Use the Poisson integral formula.

8. Let $u(x, y)$, $v(x, y)$ be continuously differentiable as functions of (x, y) in a domain Ω. Let $f(z) = u(z) + iv(z)$. Suppose that for every $z_0 \in \Omega$ there is an r_0 (depending on z_0) such that
$$\int_{|z-z_0|=r} f(z)dz = 0,$$
for all r with $r < r_0$. Prove that f is analytic in Ω. Hint: Show that f satisfies the Cauchy-Riemann equations in Ω.

9. Compute
$$\int_{|z|=4} \frac{\sin z}{z^2} dz.$$

10. Let $D_2 = \{z : |z| < 2\}$ and $I = \{x \in \mathbf{R} : -1 \leq x \leq 1\}$. Find a bounded harmonic function u, defined in $D_2 - I$ such that u does not extend to a harmonic function defined in all of D_2.

11. Suppose f is analytic on $D = \{|z| < 1\}$ and $f(0) = 0$. Prove that
$$\sum f(z^n)$$
converges uniformly on compact subsets of D.

12. Let a_k be a sequence of distinct complex numbers such that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{|a_k|}$ converges.
Let $A = \{a_k : k = 1, \ldots, \infty\}$. Prove that
$$\sum_{k=1}^{\infty} \frac{1}{z - a_k}$$
converges to an analytic function on $\mathbb{C} - A$.

13. Let f and g be entire functions so that satisfy $f^2 + g^2 = 1$. Prove that there is an entire function h so that $f = \cos(h), g = \sin(h)$.

14. Find a function, $h(x, y)$, harmonic in $\{x > 0,\ y > 0\}$, such that
$$h(x, y) = \begin{cases} 0 & \text{if } 0 < x < 2, y = 0, \\ 1 & \text{if } x > 2, y = 0, \\ 2 & \text{if } x = 0, y > 0 \end{cases}$$

15. Suppose that u is harmonic on all of $\mathbb{C}$ and $u \geq 0$. Prove that u is constant.

16. Suppose f is analytic on $H = \{z = x + iy : y > 0\}$ and suppose $|f(z)| \leq 1$ on H and $f(i) = 0$. Prove
$$|f(z)| \leq \left| \frac{z - i}{z + i} \right|.$$

17. Compute

$$\int_0^\infty \frac{1-\cos x}{x^2}dx.$$

18. Let f be a non-constant analytic function on the connected open set W. Let $Z = \{z : f(z) = 0\}$. Prove that $W - Z$ is connected.

19. Find the radius of convergence of

$$\sum \frac{n^n}{n!} z^{2n}.$$

20. Suppose $f \in \mathcal{O}(0 < |z-a| < \epsilon)$ and that $\mathrm{Re}(f)$ is bounded. Prove that a is a removable singularity.

21. Let f be a non-constant analytic function defined on $\{|z| < 1\}$ such that $\mathrm{Re}(f(z)) \geq 0$.

    (a) Prove that $\mathrm{Re}(f(z)) > 0$.

    (b) Suppose $f(0) = 1$. Prove that

    $$\frac{1-|z|}{1+|z|} \leq |f(z)| \leq \frac{1+|z|}{1-|z|}.$$

22. Suppose f is analytic on the connected open set W. Let $\{|z-z_0| \leq a\} \subset W$.

    (a) Prove that

    $$f(z_0) = \frac{1}{\pi a^2}\int_{|z-z_0|\leq a} f(z)dA.$$

    (b) Suppose f is not constant on W. Prove that

    $$|f(z_0)| < \frac{1}{\pi a^2}\int_{|z-z_0|\leq a} |f(z)|dA \text{ (strict inequality)}.$$

23. Suppose f and g are analytic on a connected open set Ω. You might want to use the previous problem on this problem.

    (a) If $|f(z)| + |g(z)|$ is constant, then both f and g are constant.

    (b) If $|f(z)| + |g(z)|$ assumes a local maximum in Ω, then f and g are constant.

24. Prove that $\sum_1^\infty \frac{\sin nz}{2^n}$ represents an analytic function on $|\mathrm{Im}(z)| < \log 2$.

25. (a) Prove that the series

$$\sum_{1}^{\infty} 2^{-n^2} z^{2^n}$$

converges uniformly on $|z| \leq 1$.

(b) Prove that the radius of convergence of the series is 1.

26. Let f be analytic on the connected open set W. Suppose $\{z : |z-a| \leq d\} \subset W$ and suppose f is real on $\{z : |z-a| = d\}$. Prove that f is constant in W.

27. Gamelin, §VI.2, # 13.

Suppose f has an isolated singularity at a and $Re(f(z))$ is bounded on $0 < |z-a| < \epsilon$. Prove that the singularity is removable.

Suppose $f(z) = u(z) + iv(z)$ is entire and $|u(z)| > |v(z)|$ for all z. Prove that f is constant.

Suppose f is analytic in $\{0 < |z| < r\}$ for some $r > 0$. Suppose also that $|f(z)| < |z|^{-1+\epsilon}$ in $\{0 < |z| < \delta\}$, where $\epsilon > 0$. Prove that f has a removable singularity at 0.

Let $\sum_{n=-\infty}^{\infty} a_n z^n$ be the of Laurent series of $\dfrac{1}{\sin(z)}$ in $|z| < \pi$. Prove that $a_n = 0$ if $n < -1$ and $a_n = 0$ if n is even. Compute a_{-1} and a_1.

There may be homework problems or example problems from the text on the final. Don't forget previous sample problems.